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A STATISTICAL INVESTIGATION OF THE FORCED RESPONSE OF FINITE, NEARLY PERIODIC ASSEMBLIES

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1. INTRODUCTION

Parameter variations in structures are unavoidable due to manufacturing tolerances, assembly defects, aging, etc. Since these imperfections are frequently small, in most systems they only produce a small perturbation in the dynamical response; hence for ease of modelling and analysis they are frequently discarded. However, small irregularities may have a drastic impact on a special class of engineering known as *periodic structures*, which consist of identical bays connected one to another in an identical manner. The dynamics of periodic structures have been shown to be highly sensitive to periodicity-breaking disorder. Namely, under conditions of weak coupling between bays, small random disorder may give rise to the occurrence of *normal mode localization* [1–5].

Mode localization is characterized by an asymptotic exponential decay of the vibration amplitude along the randomly disordered periodic structure [6–8]. When analyzing this amplitude decay, it is important that one not only examines the ensemble averages, but also the probability distribution of the rate of decay. Surprisingly, the statistics of the exponential decay rate of the vibration amplitude—the so-called localization factor—has received only scant attention in the literature. Recently, Cha and Morganti [9] calculated the statistics of the localization factor in a nearly periodic structure by utilizing probabilistic perturbation methods. They postulated the probability distributions of the localization factor for the cases of weak and strong localization, and verified the results with numerical simulations. However, their study was restricted to chains of single-degree-of-freedom bays. Castanier and Pierre [10] recently calculated localization factors as Lyapunov exponents [4, 11], and examined their statistics numerically. However, they used the covariance of the Lyapunov exponents as a possible indicator of conversion among the various wave types, and not as a means of reaching conclusions about the typical rate of exponential decay.

This note is an extension of reference [9] to more elaborate periodic structures consisting of multi-mode bays. These systems feature several frequency passbands, and thus are representative of many periodic engineering systems, e.g., bladed-disk assemblies.

2. LOCALIZATION FACTORS

2.1. *Equations of motion*

Consider the undamped assembly of N, mono-coupled, multi-mode, nearly identical bays shown in Figure 1. Each bay is coupled to its adjacent neighbors through linear springs of stiffness k_s located at $x = x_c$. To study vibration transmission, the system is

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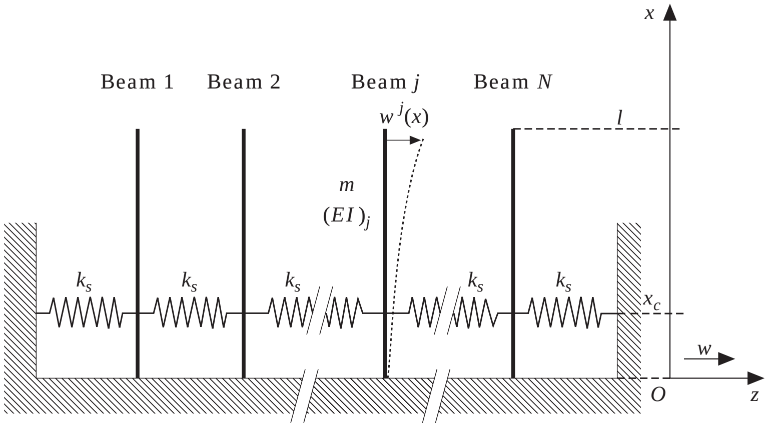

Figure 1. An assembly of coupled, multi-mode bays (beams).

excited at one end by a simple harmonic force of frequency, $\bar{\omega}$, and the steady state response at its other end is examined. The motion of each (uncoupled) bay is written in terms of its normal mode coordinates (corresponding to the modes of a fixed–free beam in Figure 1). Applying component mode analysis, the governing equations of motion for the disordered assembly are [8]

$$([K] - \bar{\omega}^2[I]) \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \vdots \\ \boldsymbol{\eta}_i \\ \vdots \\ \boldsymbol{\eta}_N \end{bmatrix} = \begin{bmatrix} \dfrac{F_0}{\mathscr{M}} \boldsymbol{\phi}_f \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \tag{1}$$

where $\boldsymbol{\eta}_i$ is the M-vector of modal co-ordinate amplitudes for the ith bay, M is the number of modes for each bay, $\boldsymbol{\phi}_f$ is the M-vector of modal deflections at the forcing location, x_f, F_0 is the forcing amplitude, $\mathscr{M}$ is the modal mass, $[I]$ is the identity matrix, and $[K]$ is the $NM \times NM$ block tridiagonal matrix

$$[K] = \begin{bmatrix} \ddots & \ddots & [0] & \cdots & [0] \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ [0] & -[\bar{R}][\Lambda] & [D_i] & -[\bar{R}][\Lambda] & [0] \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ [0] & \cdots & [0] & \ddots & \ddots \end{bmatrix}, \tag{2}$$

where

$$[A] = \boldsymbol{\phi}_c \boldsymbol{\phi}_c^{\mathrm{T}}, \qquad [D_i] = [\Lambda](1 + d\lambda_i) + 2[\bar{R}]\boldsymbol{\phi}_c \boldsymbol{\phi}_c^{\mathrm{T}}. \tag{3}$$

In equation (3), $[\bar{R}] = \bar{R}[I]$ ($\bar{R} = k_s/\mathscr{M}$) is the $M \times M$ diagonal matrix of the coupling frequency, $[\Lambda] = [\mathrm{diag}(\bar{\lambda}_p)]$ is the $M \times M$ diagonal matrix of the eigenvalues of the nominal bay (the squares of the natural frequencies of the uncoupled bays), $d\lambda_i$ is the dimensionless disorder parameter for the ith bay, and $\boldsymbol{\phi}_c$ is the M-vector of modal deflections at the constraint location, x_c. The disorders, $d\lambda_i$, are assumed to be independent, identically and uniformly distributed random variables of mean 0 and standard deviation σ. For a perfectly periodic system, $d\lambda_i = 0$.

While the results derived herein are applicable to any mono-coupled periodic structure, all of the numerical calculations are performed on an assembly of cantilever Euler–Bernoulli beams (see Figure 1). Disorder is assumed to originate from discrepancies among the flexural rigidities of the beams, $(EI)_i$. Thus, the modal mass is $\mathscr{M} = ml$, and the component modes, $\phi_p(x)$, are the eigenfunctions of a cantilever beam. The equations of motion (1) are non-dimensionalized by dividing by EI/ml^4 to introduce the dimensionless parameters $R = \bar{R}/(EI/ml^4) = k_s/(EI/l)^3$, $\lambda = \bar{\lambda}/(EI/ml^4)$ and $\omega^2 = \bar{\omega}^2/(EI/ml^4)$.

2.2. *Monte Carlo simulations*

The localization factor, γ, is the asymptotic rate of exponential amplitude decay per bay, obtained by letting the number of bays N go to infinity [7, 8]. Numerically, however, γ is calculated by taking the average, over a large number of realizations r, of the decay rate γ_N for disordered assemblies of finite size N.

Two Monte Carlo simulation schemes were developed to approximate the localization factor [7, 8]. For strong coupling (when the localization is weak), a wave formulation is adopted, in which an N-bay disordered chain is embedded in an otherwise ordered infinite system. The N-bay decay rate is given by

$$\gamma_N = -\frac{1}{N}\ln|\tau_N|, \tag{4}$$

where τ_N, the transmission coefficient for the disordered segment, is the inverse of the (1, 1) term of the wave transfer matrix, obtained by multiplying transfer matrices for the N bays and applying the appropriate similarity transformations [4]. Taking the average of γ_N over many disordered segments allows one to obtain an approximation of the localization factor.

For weak coupling (when the localization is strong), a modal formulation is selected, which considers an N-bay disordered assembly with fixed–fixed end conditions. The N-bay decay rate is then given by

$$\gamma_N = -\frac{1}{N}\ln|\boldsymbol{\phi}_c^{\mathrm{T}}[F_{1N}]\boldsymbol{\phi}_f|, \tag{5}$$

where $[F_{1N}]$ denotes the $(1, N)$th $M \times M$ submatrix of $([K]-\bar{\omega}^2[I])^{-1}$, which can be obtained recursively by exploiting the tridiagonality nature of the $[K]$ matrix.

The inherent difference between the above two approaches is that the modal formulation incorporates boundary effects, while the wave approach does not. Both wave and modal formulations can be used to describe properly the type of localization—weak or strong—at hand.

3. STATISTICAL DISTRIBUTION OF THE LOCALIZATION FACTOR

3.1. *Numerical results*

To illustrate the statistical distribution of the localization factor, consider the case $R = 3{\cdot}0$ and $x_c = 1{\cdot}0$. For simulation purposes, 10 000 realizations of ten random bays each are used for the wave simulation in the first two passbands (in order to eliminate the effects of the boundary conditions when the decay due to disorder is weak), while 10 000 realizations of 50 random bays each are chosen for the modal simulation in the third and

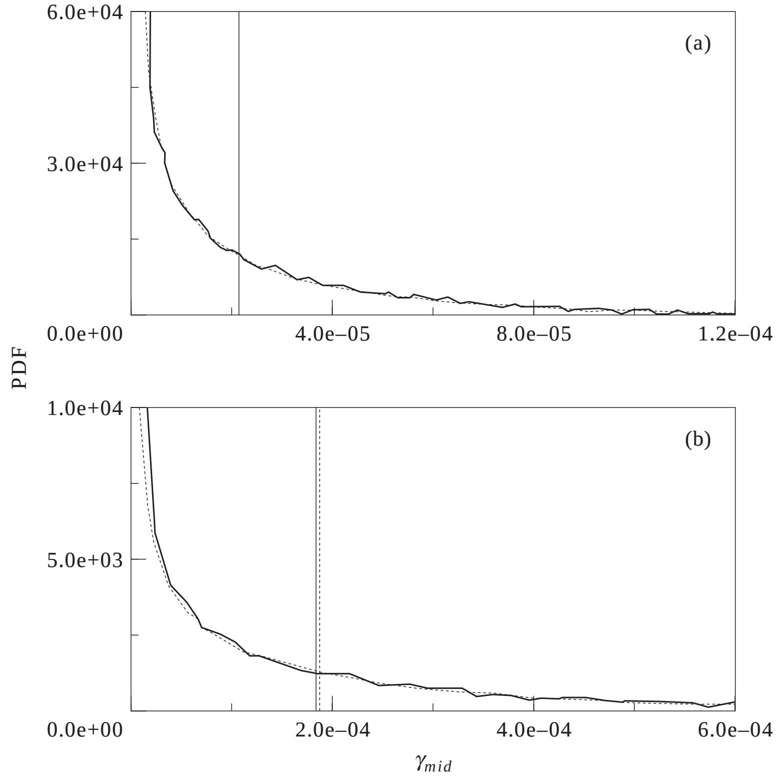

Figure 2. The probability density functions of the first mid-band localization factor for two disorder strengths by Monte Carlo wave simulation (——) and postulated distribution based on strong coupling perturbation method (· · ·). The system parameters are $R = 3{\cdot}0$, $x_c = 1{\cdot}0$, $N = 10$ and $r = 10\,000$. The solid and dotted vertical lines represent the numerical and perturbation values of $\langle \gamma_{mid} \rangle$, respectively. (a) $\sigma = 1\%$; (b) $\sigma = 3\%$.

fourth passbands. In order to simplify the analysis, only the distribution of γ at mid-band frequencies is considered.

In Figure 2 is depicted the statistical distribution of the first mid-band localization factor, γ_{mid}. Note that in the first passband localization is weak, and that the distribution is of exponential type for a disorder strength up to 3%. Since the distribution exhibits rightward skewness with a large tail, the expected value alone does not reveal the possibility of a much higher γ that can occur in the tail within the ensemble. In addition, notice that γ's less than the mean value possess higher probabilities of occurrence. Thus, for such *weak localization*, the behavior of a typical system may deviate substantially from that predicted by the mean. The average of γ_{mid} can be a misleading indicator of the expected rate of decay in any single realization of the random medium, and care must be taken when using this value to predict behavior. In the second passband, shown in Figure 3, the numerically obtained distribution is again of exponential type for 1% disorder. For a disorder of 3%, however, the exponential distribution characteristics begin to disappear, and the transition from an exponential to a normal probability distribution can be noted. It can be shown that as the disorder strength increases, the distribution tends to a normal one, such that the mean of γ_{mid} becomes an order of magnitude larger than its standard deviation. Thus, for a sufficiently large disorder, the behavior of a typical system can be predicted by that of the mean response.

The simulated probability density functions of the mid-band localization factor in the third and fourth passbands are illustrated in Figure 4. Observe that the localization is strong, and that the distribution of γ_{mid} is normal with a given variance. This implies that the average response is indeed characteristic of how a typical disordered system will behave. Hence, at sufficiently high frequencies, typical behavior can be predicted by the mean localization factor, even for a very weakly disordered structure.

3.2. *Standard deviation of γ for weak localization*

Using statistical perturbation methods, the standard deviations of γ, as well as its mean, can be approximated. For strong coupling, the approximation of the mean mid-band localization factor is [8].

$$\langle \gamma^s_{mid} \rangle \simeq \frac{\sigma^2 \alpha^2_{p'}}{8R^2 \phi^4_p(x_c)/\lambda^2_p}, \tag{6}$$

where the superscript "s" denotes the strong coupling case. After lengthy algebra, the corresponding standard deviation is found to be

$$\sigma_{\gamma^s_{mid}} \simeq \frac{\sigma^2 \alpha^2_{p'}}{4\sqrt{2}R^2 \phi^4_p(x_c)/\lambda^2_p}. \tag{7}$$

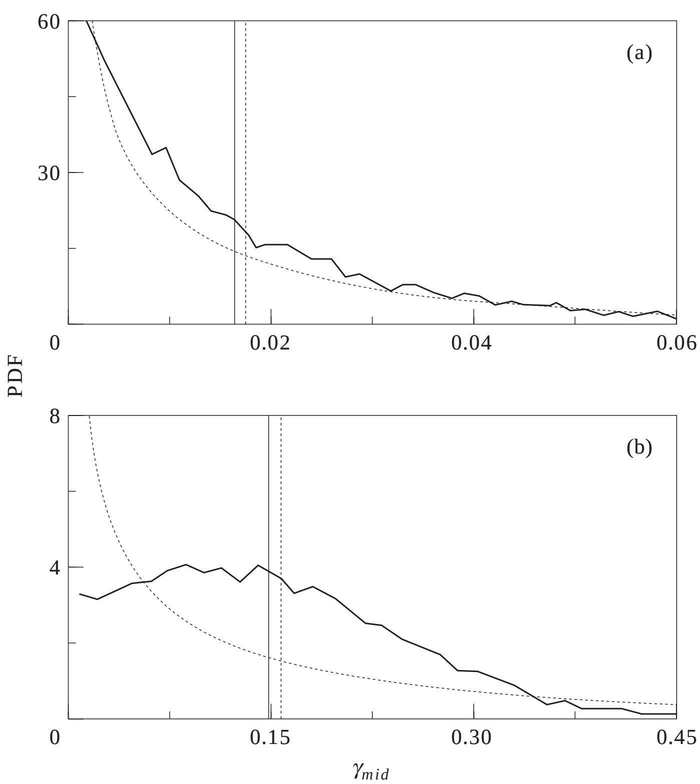

Figure 3. The probability density functions of the second mid-band localization factor for two disorder strengths by Monte Carlo wave simulation (——) and postulated distribution based on strong coupling perturbation method (· · ·). The system parameters are $R = 3{\cdot}0$, $x_c = 1{\cdot}0$, $N = 10$ and $r = 10\,000$. The solid and dotted vertical lines represent the numerical and perturbation values of $\langle \gamma_{mid} \rangle$, respectively. (a) $\sigma = 1\%$; (b) $\sigma = 3\%$.

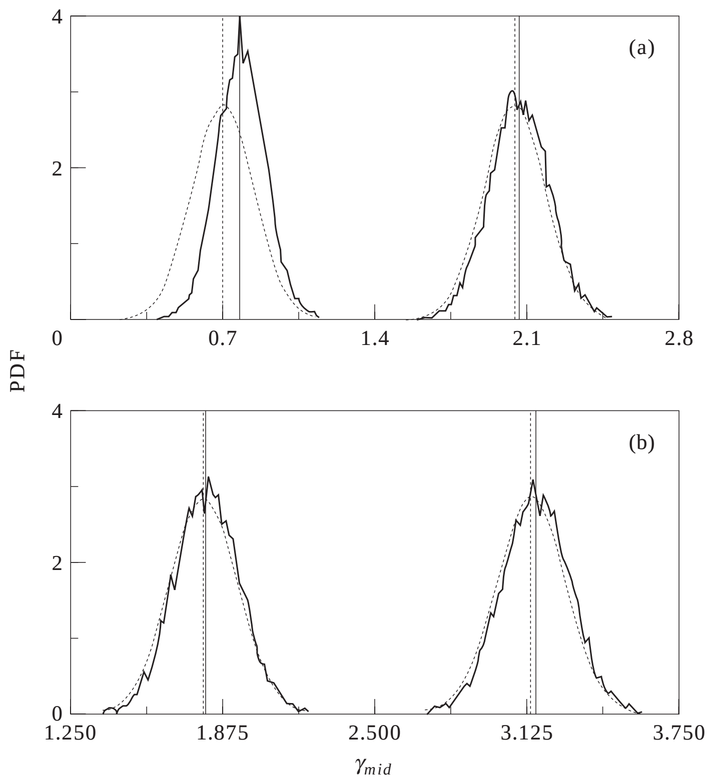

Figure 4. The probability density functions of the third and fourth mid-band localization factor for two disorder strengths by Monte Carlo modal simulation (——) and postulated distribution base on weak coupling perturbation method (· · ·). The system parameters are $R = 3{\cdot}0$, $x_c = 1{\cdot}0$, $N = 50$ and $r = 10\,000$. The solid and dotted vertical lines represent the numerical and perturbation values of $\langle \gamma_{mid} \rangle$, respectively. (a) $\sigma = 1\%$; (b) $\sigma = 3\%$.

Comparing equations (6) and (7), we note that for this case of weak localization:

$$\sigma_{\gamma^s_{mid}} = \sqrt{2} \langle \gamma^s_{mid} \rangle. \tag{8}$$

The good agreement between the simulated and analytical mean of the first mid-band localization factor in cases of weak localization is displayed in Figure 2. The distribution of γ_{mid} appears to be exponential, and since equation (8) tells us that the mean and standard deviation are proportional, we speculate that the actual continuous probability distribution is of gamma type. From reference [12], the probability density function for a gamma distribution is characterized by parameters α and β ($\beta > 0$) and given by

$$f(x) = \begin{cases} \dfrac{\beta \mathrm{e}^{-\beta x} (\beta x)^{\alpha - 1}}{\Gamma(\alpha)}, & x \geqslant 0, \\ 0, & x < 0, \end{cases} \tag{9}$$

where $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-x} x^{\alpha-1}\,\mathrm{d}x$ is the gamma function. The mean and variance of the gamma distribution are α/β and α/β^2, respectively.

To validate our speculation, the postulated gamma distribution of the first mid-band localization factor is plotted in Figure 2. The mean and standard deviation, derived using the strong coupling perturbation method and given by equations (6) and (7), are used as the parameters for the postulated gamma distribution in equation (9). Note the excellent agreement between the simulated and the postulated probability density function for a disorder up to 3%. In the second passband (see Figure 3), the agreement between the postulated and simulated distributions is good for a disorder of 1%. Not surprisingly, for a 3% disorder, the postulated results no longer track the simulation solution, since the ratio of disorder to coupling no longer satisfy the constraint under which equation (6) was derived, namely a small disorder to coupling ratio. However, the analytical mean still approximates the numerical value fairly well. It is interesting to note that if the *simulation* mean and standard deviation were used as the parameters for the gamma distribution, then the postulated probability density function tracks the simulation results reasonably well, as shown in Figure 5.

3.3. *Standard deviation of γ for strong localization*

For higher passbands, the coupling between bays decreases, the frequency bands become widely separated, and there is limited interaction between the mode groups. Hence the bays vibrate primarily in their uncoupled modes, and a single mode analysis is often sufficient to capture the bahavior of the system. To the first order approximation, the average of the mid-band localization factor is [8]

$$\langle \gamma_{mid}^{w} \rangle \simeq \ln\left| \frac{\sigma}{R\phi_p^2(x_c)/\lambda_p} \right| + \ln\sqrt{3} - 1, \tag{10}$$

where the superscript "w" denotes the weak coupling case. The standard deviation of γ_{mid} is found to be

$$\sigma_{\gamma_{mid}^{w}} \simeq 1/\sqrt{N}. \tag{11}$$

Note that for this case of weak coupling or strong localization, the standard deviation equals the inverse of the square root of the distance from the excited end, the same result that can be inferred from reference [13]. While equation (11) is derived for a mid-band excitation, it can be shown to remain valid for any excitation frequency.

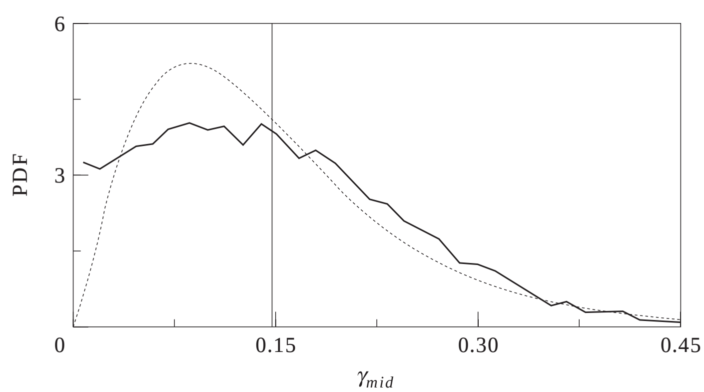

Figure 5. The probability density function of the second mid-band localization factor for $\sigma = 3\%$ by Monte Carlo wave simulation (——) and postulated distribution based on strong coupling perturbation method (· · ·), obtained using the simulated mean and standard deviation. The system parameters are $R = 3{\cdot}0$, $x_c = 1{\cdot}0$, $N = 10$ and $r = 10\,000$.

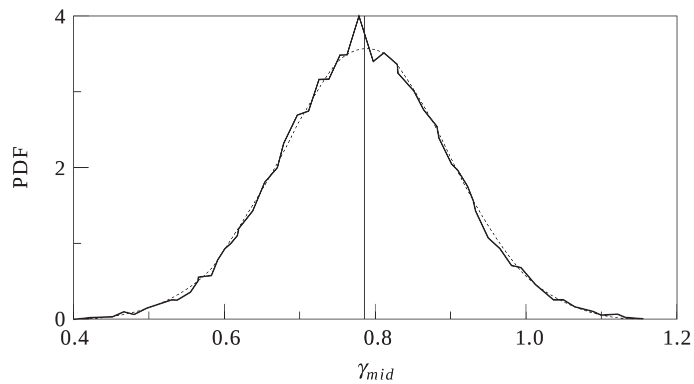

Figure 6. The probability density function of the third mid-band localization factor for $\sigma = 1\%$ by Monte Carlo modal simulation (——) and postulated distribution based on weak coupling perturbation method (· · ·), obtained using the simulated mean and standard deviation. The system parameters are $R = 3{\cdot}0$, $x_c = 1{\cdot}0$, $N = 50$ and $r = 10\,000$.

Inspection of Figure 4 shows that the simulated probability density function of the mid-band localization factor appears to be normal. From reference [12], the probability density function of a normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi v}}\, e^{-(x-\mu)^2/2v^2}, \qquad \infty < x < \infty, \tag{12}$$

where the mean and standard deviation are μ and v, respectively.

Using the mean and the standard deviation given by equations (10) and (11), respectively, and derived with the weak coupling perturbation method, a normal probability density function can be postulated for γ_{mid}. It is shown in Figure 4 that in the third passband, for a disorder of 1%, the agreement between the analytical and the simulated results is poor. However, if we use the *simulation* mean and standard deviation as the parameters for our postulated normal distribution, then the agreement between the simulated and postulated probability density function becomes excellent (see Figure 6). For a disorder of 3%, note the very good agreement between the simulated and postulated density curves in Figure 4. This is due to the fact that the perturbation approximation is valid for large ratios of disorder to coupling. The mid-band localization factor as obtained from the modal simulations is shown in Table 1. Note that, in a given passband, γ_{mid}

TABLE 1

The simulated mid-band localization factor and its standard deviation for various disorder strengths. Obtained from Monte Carlo modal simulations with $r = 10\,000$, $N = 50$, $R = 3{\cdot}0$, *and* $x_c = 1{\cdot}0$. *The standard deviation predicted by the perturbation method for weak coupling is* $1/\sqrt{50} = 0{\cdot}1414$.

Passband number	Disorder strength (%)	γ_{mid}	$\sigma_{\gamma_{mid}}$
Third	1	0·7871	0·1112
	2	1·4206	0·1295
	3	1·8159	0·1349
Fourth	1	2·0699	0·1360
	2	2·7605	0·1389
	3	3·1678	0·1398

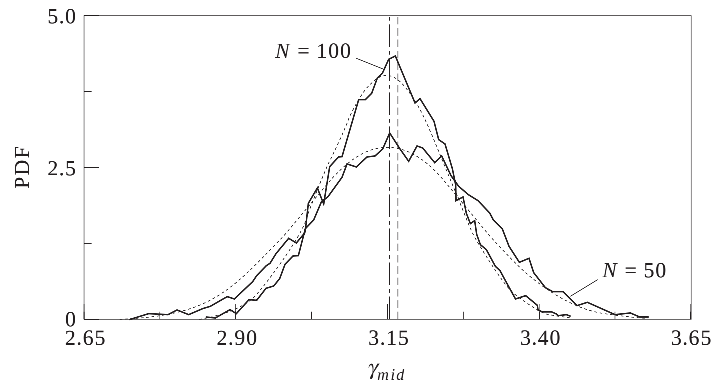

Figure 7. The probability density functions of the fourth mid-band localization factor, γ_{mid}, for $N = 50$ (– – –) and $N = 100$ (– · –) by Monte Carlo modal simulation (——) and postulated distribution based on weak coupling perturbation method (· · ·). The system parameters are $R = 3{\cdot}0$, $x_c = 1{\cdot}0$, $\sigma = 3\%$ and $r = 10\,000$. The vertical lines denote $\langle \gamma_{mid} \rangle$ for $N = 50$ and $N = 100$.

increases with the disorder strength, and for a given disorder, it also increases with the passband number. Finally, note that the simulated standard deviation of γ_{mid} tends to its analytical prediction as localization becomes stronger.

The simulated probability distribution of γ_{mid} for assemblies of 50 and 100 components is shown in Figure 7. Again, 10 000 realizations are used. Note that the variance for $N = 100$ is indeed smaller than for $N = 50$. Thus, as the number of bays becomes large, the standard deviation becomes small compared to its mean, making the mean a good predictor of typical response. Observe the good agreement between simulated and postulated probability density functions.

3.4. *Effect of constraint location*

In reference [8] it was shown that the degree of localization depends on the ratio of disorder to coupling, $R\phi_p^2(x_c)/\lambda_p$, and that the constraint location affects the localization factor in a given passband through the modal deflection at the constraint, $\phi_p(x_c)$. The effect of the constraint location on the probability density function of γ_{mid} is illustrated in Figure 8. The constraint location x_c is very close to the node of the second component mode, the coupling stiffness R is large, and the disorder strength σ very small, yielding small localization factors in the first and third passbands, but large γ_{mid} in the second passband, where the component beams are nearly decoupled. Not surprisingly, the probability distribution of γ_{mid} also reveals the same trend. In the first and third passbands, where the localization factors are small, the probability density function of γ_{mid} is of exponential type, indicating weak localization. In the second passband where the localization is severe, γ_{mid} is normally distributed. Note the excellent agreement between the simulated and postulated probability density functions in all three passbands.

4. CONCLUSIONS

Due to the random nature of the irregularities, the rate of exponential amplitude decay for a finitely long, disordered chain is statistically distributed. Depending on the ratio of disorder to coupling, the fluctuation of the localization factor about its mean can be large. For small disorder to coupling ratios, localization is weak and the localization factor approximately features a *gamma distribution*, with its standard deviation being $\sqrt{2}$ times its expected value. This implies that the response of an arbitrary realization of the

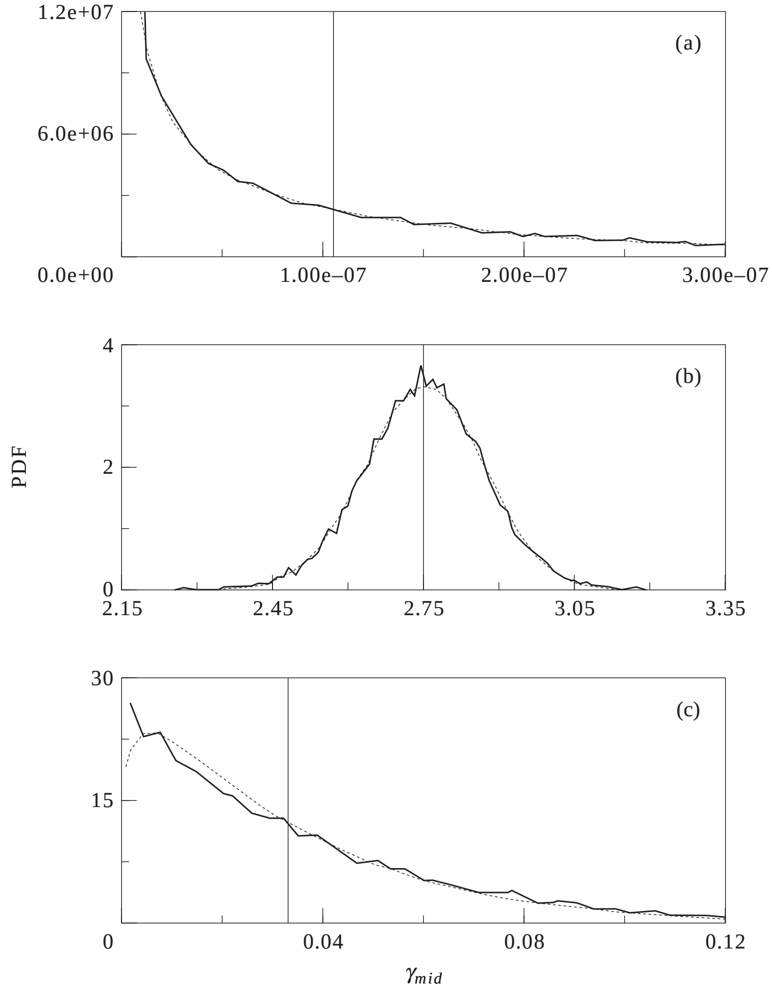

Figure 8. The probability density functions of the first three mid-band localization factors ((a), (b) and (c)) by Monte Carlo wave and modal simulations (——) and postulated distribution based on strong and weak coupling perturbation methods (· · ·). The system parameters are $R = 15{\cdot}0$, $x_c = 0{\cdot}78$, $r = 10\,000$ and $\sigma = 0{\cdot}2\%$. For the wave simulation, $N = 10$; for the modal simulation, $N = 50$. The vertical line represents the simulated $\langle \gamma_{mid} \rangle$.

disordered system may deviate dramatically from predicted mean behavior. For large disorder to coupling ratios, however, localization is strong and the localization factor is approximately *normally distributed*, with its standard deviation being proportional to the square root of chain size. Thus, for a system with many sites, the mean behavior accurately depicts the response of a typical assembly.

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